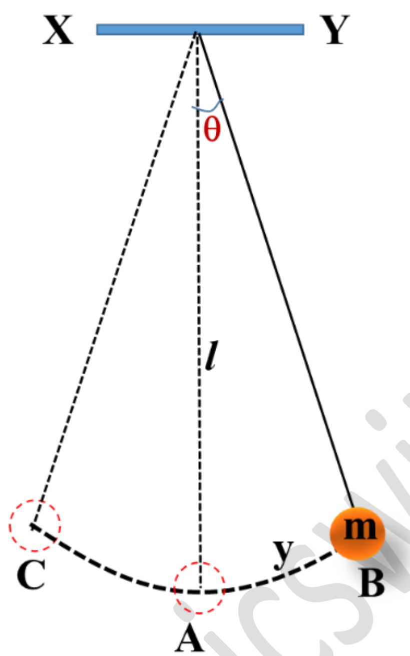
Simple Harmonic Motion

Harmonic Motion/Periodic Motion:

A type of motion in which an object repeats its path in regular internal of time is called harmonic motion.

Simple Harmonic Motion (S.H.M.)

Let us consider an object of mass 'm' attached to a string is suspended from a rigid support XY. The object is



displaced from position A to B through a small displacement (y).

Experimentally, it is found that the restoring force, 'F' is directly proportional to displacement (y).

i.e. $F \propto y$

$F = -ky$

[The -ve sign shows that force is restoring and acts in the direction opposite to displacement.]

or, $ma = -ky$

or, $a = \frac{-k}{m} y$

or, $a = -k'y \left(\text{Here } \frac{k}{m} = k' \text{ is a new constant} \right)$

$\therefore a \propto y$

Thus **S.H.M. is a type of motion in which an object moves to and fro about mean position and acceleration is directly proportional to displacement and directed towards the mean position.**

Circle as a reference of S.H.M.

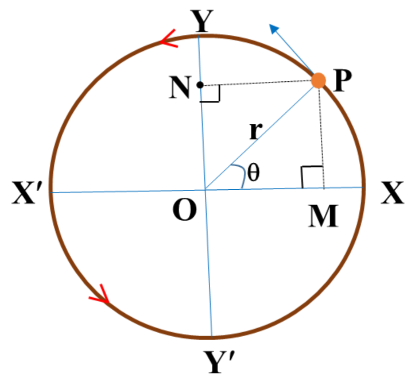


Fig: S.H.M. of an object in vertical diameter YOY′

Let us consider an object of mass 'm' is moving with angular velocity 'ω' on a circular path of radius 'r' Suppose after time 't' the position of the object is at P. Let us draw perpendicular (PN) from point 'P' to the vertical line (Y-axis). Then we can say that the foot of the perpendicular drawn from the body moving in a circular path executes in simple harmonic motion with amplitude 'r' about the mean position 'O' in vertical diameter YOY′.

Characteristics of S.H.M

(1) Displacement (y): It is defined as the shortest distance from mean position. In the given fig. 'ON =y' is displacement from mean position 'O'.

From figure,

$\text{Sin}\theta = \frac{y}{r}$ $\quad \because \text{Sin}\theta = \frac{p}{h} = \frac{PM}{OP}$

Or, $y = r\sin\theta$

$\because \omega = \frac{\theta}{t} \quad \rightarrow \theta = \omega t$

$\therefore$ **$y = r\sin\omega t$**

(2) Amplitude (r): It is defined as the maximum value of displacement.

We have, displacement, $y = r\sin\theta$

For maximum displacement, $\sin\theta = 1$

$\therefore$ **Amplitude = $y_{max} = r$**

(3) Velocity (v): It is defined as the rate of change of displacement.

i.e. velocity, $v = \frac{dy}{dt}$

Or, $v = \frac{d}{dt}(r\sin\omega t) \quad \because y = r\sin\omega t$

Or, $v = r\frac{d}{dt}(\sin\omega t)$

Or, $v = r\omega\cos\omega t$

Or, $v = r\omega\sqrt{1-\sin^2\omega t}$

Or, $v = r\omega\sqrt{1-\frac{y^2}{r^2}} \quad (\because y = r\sin\omega t)$

Or, $v = \frac{r\omega \sqrt{r^2 - y^2}}{r}$

$\therefore$ $\mathbf{v = \omega\sqrt{r^2 - y^2}}$

Special cases

(i) At mean position, displacement, y = 0

So, velocity, $v = \omega\sqrt{r^2 - 0^2}$

$\therefore v = \omega r$ (maximum velocity)

(ii) At extreme positions, displacement, y = r

$v = \omega\sqrt{r^2 - r^2} = 0$

v = 0 (minimum velocity)

(4) Acceleration (a): It is defined as the rate of change of velocity.

i.e. $a = \frac{dv}{dt}$

Or, $a = \frac{d}{dt}\left(\frac{dy}{dt}\right)$

Or, $a = \frac{d}{dt}[\frac{d}{dt}(r \sin\omega t)]$

Or, $a = \frac{d}{dt}(r\omega \cos\omega t)$

Or, $a = r\omega \frac{d}{dt}(\cos \omega t)$

Or, $a = r\omega(-\omega \sin\omega t)$

Or, $a = -\omega^2 r \sin\omega t$

Or, $\mathbf{a = -\omega^2 y}$ $\because y = r \sin\omega t$

Special Cases:

(i) At mean position, displacement, y = 0

$a = -\omega^2 0 = 0$ (minimum)

(ii) At extreme positions, displacement y = r

$a = -\omega^2 r$ (maximum)

(5) Time Period (T): Time period or an object in S.H.M. is defined as the time to complete one oscillation.

Time Period, $T = \frac{2\pi}{\omega}$

$\because a = \omega^2 y$ So $\omega = \sqrt{\frac{a}{y}}$

$T = 2\pi\sqrt{\frac{y}{a}}$

$\mathbf{T = 2\pi\sqrt{\frac{displacement}{acceleration}}}$

(6) Frequency (f):

It is defined as the no. of complete oscillation made for second.

Frequency $f = \frac{1}{T}$

$\mathbf{f = \frac{1}{2\pi}\sqrt{\frac{acceleration}{displacement}}}$

(7) Phase Angle (ϕ): It is defined as the angle made by initial position of the object with the mean position.
Note: In displacement equation

$y = r \sin(\omega t + \phi)$ where, y is displacement, r is amplitude, ϕ is phase angle,

Time period, $T = \frac{2\pi}{\omega}$

Velocity, $V = \frac{dy}{dt}$

Acceleration, $a = \frac{dV}{dt} = \frac{d^2y}{dt^2}$

Simple Pendulum:

A heavy point mass suspended from a rigid support with the help of weightless, inextensible and perfectly flexible string which can oscillate freely about mean position is called simple pendulum.

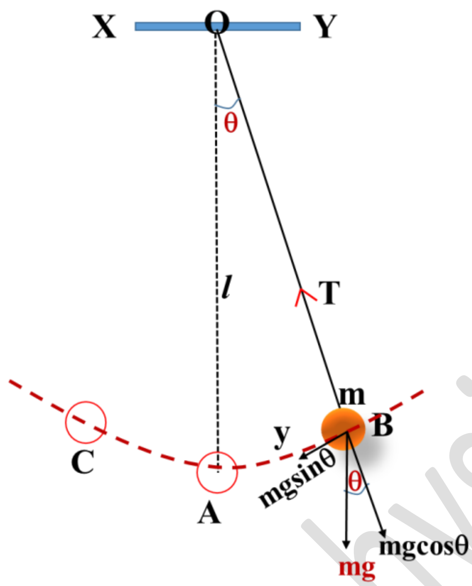


Fig: Simple pendulum executing S.H.M

Let us consider a simple pendulum of mass 'm' and effective length 'l' is oscillating about the mean position 'A'. Let at any instant the bob is at position 'B' where weight 'mg' acts vertically downward. Resolving mg into its constituent components we get, mgcosθ and mgsinθ. The component mgcosθ is balanced by Tension 'T' and mgsinθ provides restoring force.

i.e. $F = -mg\sin\theta$ [-ve sign indicates that the force is restoring]

or, $ma = -mg\sin\theta$ where 'a' is acceleration of the system

$a = -g\sin\theta$

for small angle θ, $\sin\theta \approx \theta$

$\therefore a = -g\theta$

also, $\theta = \frac{\text{arc length}}{\text{radius}} = \frac{\overset{\frown}{AB}}{l} = \frac{AB}{l} = \frac{y}{l}$

$\therefore a = -g\frac{y}{l}$ .... (1)

This shows that $a \propto y$

Since acceleration is directly proportional to displacement so the <u>motion of simple pendulum is simple harmonic motion.</u>

Time Period (T):
For simple harmonic motion acceleration is given by
$a = -\omega^2 y$ ....(2)
from eqn (1) & (2)

$$-\omega^2 y = -g \frac{y}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{g}{l}}$$

$$\therefore T = 2\pi\sqrt{\frac{l}{g}}$$ is the required expression for time period of simple pendulum.

Second pendulum and its effective length:

A simple pendulum for which time period is 2 sec is known as a second pendulum.

For simple pendulum

$$T = 2\pi\sqrt{\frac{l}{g}}$$

For second pendulum, T = 2 sec, then above equation becomes,

$$2 = 2\pi\sqrt{\frac{l}{g}}$$

Squaring both sides

$$4 = 4\pi^2 \frac{l}{g}$$

$$l = \frac{g}{\pi^2} = \frac{9.8}{\pi^2} = 0.99\text{m}$$

∴ The effective length of second pendulum is 0.993 m.

Short Question-Answer:

Q.1. A SHM is represented by $y = r \sin(\omega t + \phi)$ in usual notation. Find its acceleration.

Ans: The displacement is given by, $y = r \sin(\omega t + \phi)$
Acceleration (a) is defined as the rate of change of velocity w.r.t. time.

i.e. $a = \frac{dv}{dt}$

Or, $a = \frac{d}{dt}\left(\frac{dy}{dt}\right)$

Or, $a = \frac{d}{dt}\left[\frac{d}{dt} r \sin(\omega t + \phi)\right]$

Or, $a = \frac{d}{dt}[r\,\omega\cos(\omega t + \phi)]$

Or, $a = r\omega \frac{d}{dt}[\cos(\omega t + \phi)]$

Or, $a = r\omega\,[-\omega\sin(\omega t + \phi)]$

Or, $a = -\omega^2 r\sin(\omega t + \phi)$

Or, $\mathbf{a = -\omega^2 y}$ $\because y = r\sin(\omega t + \phi)$

Q.2. How does the frequency of vibration of simple pendulum is related with its length? Hence estimate the frequency of a second's pendulum.

Solution:

The time period of simple pendulum is given by

$T = 2\pi\sqrt{\frac{l}{g}}$ where, l is effective length and g is acceleration due to gravity.

Frequency, $f = \frac{1}{T}$

$\therefore f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ is the relation between frequency and length of simple pendulum.

For second pendulum, time period, T = 2 Sec

Since, frequency, $f = \frac{1}{T}$

$\therefore f = \frac{1}{2} = 0.5$ Hz

Q.3. On what factors does the period of a simple pendulum depend?

Ans: The time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Above expression shows that the time period of simple pendulum depends upon

(i) effective length (l) as $T \propto \sqrt{l}$

and (ii) value of acceleration due to gravity (g) as $T \propto \sqrt{\frac{1}{g}}$

Q.4. A pendulum clock is taken to moon, will it gain or lose the time? Why?

The time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Above expression shows that the time period of simple pendulum depends upon
(i) effective length (l) as $T \propto \sqrt{l}$

and (ii) value of acceleration due to gravity (g) as $T \propto \sqrt{\frac{1}{g}}$

Since the acceleration due to gravity of moon is less than that of earth so time period increases and hence it loses time.

Q.5. What do you mean by second's pendulum? Find its effective length.

Ans: Second pendulum and its effective length:
A simple pendulum for which time period is 2 sec is called a second pendulum.
For simple pendulum

$$T = 2\pi\sqrt{\frac{l}{g}}$$

For second pendulum, T = 2 sec

$$\therefore 2 = 2\pi\sqrt{\frac{l}{g}}$$

Squaring both sides

$$4 = 4\pi^2\frac{l}{g}$$

$$l = \frac{g}{\pi^2} = \frac{9.8}{\pi^2} = 0.99\text{m}$$

$\therefore$ The effective length of second pendulum is 0.99m.

 # Motion of Horizontal spring-mass system on frictionless surface.

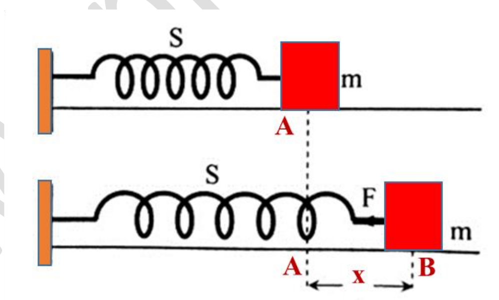


Fig: Motion of spring-mass system on horizontal plane

Let us consider a spring whose one end is fixed to rigid support and a mass 'm' is connected to another end that rests on a frictionless horizontal surface. When a small force is applied to mass to displace it from A to B through displacement 'x' then the spring-mass system moves to and fro about the mean position A.
Then from Hooke's law, the restoring force is directly proportional to displacement. i.e.

$$F \propto x$$

Or, F = -kx......... (1) Where k is called <u>force constant or spring constant</u>.
From Newton's 2nd law of motion
F = ma........ (2)
$\therefore$ ma = -kx

Or, $a = \frac{-k}{m}x$...... (3)

$\therefore$ $a \propto x$

Since, the acceleration is directly proportional to displacement.
So the motion of spring - mass system on horizontal surface is simple harmonic.

Time Period (T):

For simple harmonic motion, acceleration $a = -\omega^2 x$....... (4)
From equation (3) & (4)

$$-\omega^2 x = \frac{-k}{m}x$$

Or, $\omega^2 = \frac{k}{m}$

Or, $\omega = \sqrt{\frac{k}{m}}$

Or, $\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$

$\therefore T = 2\pi\sqrt{\frac{m}{k}}$ which is the required expression for time period.

Motion of a vertically loaded spring.

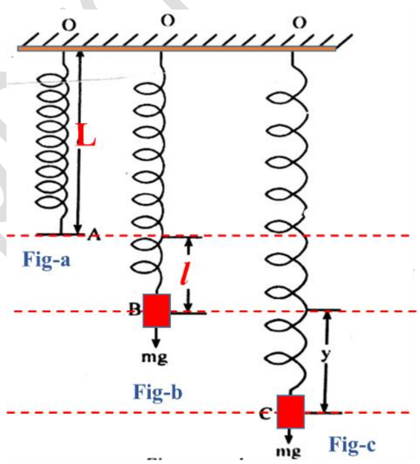


Fig: Motion of vertically loaded spring

Let us consider a spring is fixed to a rigid support and a mass 'm' is suspended to another end to displace it from position 'A' to 'B' through small distance *l*.
Then using Hooke's law in fig (b), the restoring force is directly proportional to displacement. i.e.

$F_1 \propto l$

Or, $F_1 = -kl$ Where k is called force constant or spring constant.
This restoring force balances the weight (mg)

i.e. $F_1 = mg$

$\therefore$ mg = -kl .... (1)

An additional force is then applied vertically downward to displace is further from B to C through distance 'y'. Then using Hooke's law in fig (c), the restoring force is directly proportional to displacement. i.e.

$F_2 \propto (l + y)$

$F_2 = -k\,(l + y)$

This restoring force balances the weight (mg) and gives acceleration to the system as well. i.e.

$F_2 = mg + ma$

$\therefore$ mg + ma = -k (l + y) ....(2)

Using equation (1)

$-kl + ma = -kl - ky$

Or, $ma = -ky$

Or, $a = \frac{-k}{m}y$.......... (3)

$\therefore$ $a \propto y$

Since acceleration is directly proportional to displacement, so the motion of a vertically loaded spring is simple harmonic.

Time Period (T):

For S.H.M. the acceleration is given by

$a = -\omega^2 y$.......... (4)

From (3) & (4)

$-\omega^2 y = \frac{-k}{m}y$

Or, $\omega = \sqrt{\frac{k}{m}}$

Or, $\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$

$\therefore \mathbf{T = 2\pi\sqrt{\frac{m}{k}}}$ which is the required expression for time period.

Q.3 Define simple harmonic motion. Show that motion of a vertically loaded spring is simple harmonic.

Total energy of a simple harmonic oscillator and its time period

Simple harmonic oscillator:

Simple harmonic oscillator is an object which moves to and fro about mean position and acceleration is directly proportional to displacement and directed towards the mean position.

Let us consider a body of mass 'm' moves in simple harmonic motion. The total energy of a simple harmonic oscillator is the sum of its P.E. & K.E.

i.e. T.E. = P.E. + K.E....... (1)

For S.H.M. the acceleration can be expressed as

$a = -\omega^2 y$

Then restoring force is given by

$F = ma = -m\omega^2 y$........ (2)

Let 'dy' be the small displacement then small work done 'dW' is given by

dW= -F.dy..... (3) [-ve sign indicates that the work is done against the restoring force.]

Then total work done can be obtained by integrating equation (3). i.e.

$$\int_0^y dw = \int_0^y -F dy$$

Or, $W = \int_0^y -(-m\omega^2 y)\,dy$

Or, $W = m\omega^2 \int_0^y y\,dy$

Or, $W = m\omega^2 \left[\frac{y^2}{2}\right]_0^y$

Or, $W = m\omega^2 \left[\frac{y^2}{2} - \frac{0^2}{2}\right]$

Or, $W = \frac{1}{2} m\omega^2 y^2$

According to work energy theorem this work done is stored as P.E.

$\therefore$ **P.E. =** $\mathbf{\frac{1}{2} m\omega^2 y^2}$......... (4)

This is the required expression for P.E. of simple harmonic oscillator.

Now, $K.E. = \frac{1}{2} mv^2$

Or, $K.E. = \frac{1}{2} m [\omega\sqrt{r^2 - y^2}\,]^2 \quad \because v = \omega\sqrt{r^2 - y^2}$

$\therefore$ **K.E. =** $\mathbf{\frac{1}{2} m\omega^2 (r^2 - y^2)}$ .... (5)

This is the required expression for K.E. of simple harmonic oscillator.

From (1),

T.E. = P.E. + K.E.

Or, $T.E. = \frac{1}{2} m\omega^2 y^2 + \frac{1}{2} m\omega^2 (r^2 - y^2)$

Or, $T.E. = \frac{1}{2} m\omega^2 y^2 + \frac{1}{2} m\omega^2 r^2 - \frac{1}{2} m\omega^2 y^2$

$\therefore$ **T.E. =** $\mathbf{\frac{1}{2} m\omega^2 r^2}$

This is the required expression for total energy of simple harmonic oscillator.

Special Cases:

(i) At mean position, y = 0

$P.E. = \frac{1}{2} m\omega^2 y^2 = 0$

$K.E = \frac{1}{2} m\omega^2 (r^2 - y^2) = \frac{1}{2} m\omega^2 r^2 = T.E.$

(ii) At extreme positions, y = r

$K.E = \frac{1}{2} m\omega^2 (r^2 - y^2) = 0$

$P.E. = \frac{1}{2} m\omega^2 y^2 = \frac{1}{2} m\omega^2 r^2 = T.E$

The K.E. and P.E. change with position. When one increases other decreases but the total energy of the body in S.H.M. remains constant. The variation of energy is shown in figure below

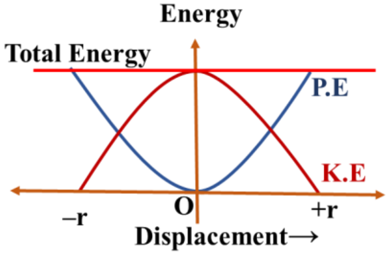


Fig: Variation of K.E. and P.E as a function of position in S.H.M

Time Period

Since, time period can be expressed as

$$T = \frac{2\pi}{\omega}$$

So $$\omega = \frac{2\pi}{T}$$

Then, $$T.E. = \frac{1}{2} m\omega^2 r^2$$

Or, $$T.E. = \frac{1}{2} m \frac{4\pi^2}{T^2} r^2$$

Or, $$T.E. = m \frac{2\pi^2}{T^2} r^2$$

Or, $$T^2 = \frac{2m\pi^2 r^2}{T.E.}$$

$$\therefore \quad \boxed{T = \pi r \sqrt{\frac{2m}{T.E.}}}$$

This is the expression for time period in terms of T.E.

Angular Simple Harmonic Motion

Angular simple harmonic motion is a type of motion in which an object rotates to and fro about a reference line and angular acceleration is directly proportional to angular displacement and directed towards the mean line. For example, a photo frame or a calendar suspended from a nail on the wall. If it is slightly pushed from its mean position and released, it makes angular oscillations.

Angular motion of balance wheel

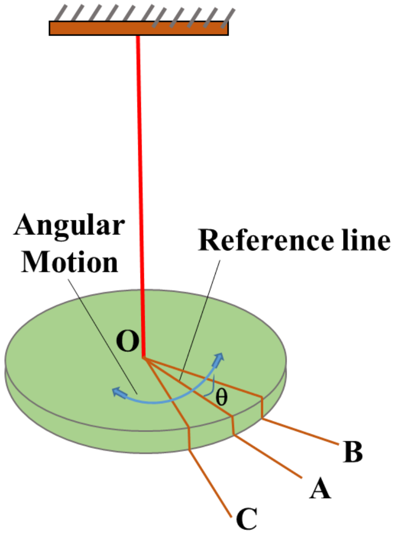


Fig: Torsional Pendulum

Let us consider a balanced wheel of moment of inertia (I) attached to a wire (or spring), suspended with the help of a rigid support, oscillates about the axis (the suspension wire) passing through its centre. When the balance wheel is rotated, the wire/spring exerts restoring torque. Let OA be the equilibrium line (mean line) as shown in figure.

Let at any instant, the wheel rotates and makes an angle θ with the equilibrium line such that $\angle AOB = \theta$

Experimentally it is found that the restoring torque τ is directly proportional to the angular displacement θ

i.e. $\tau \propto \theta$

Or, $\tau = -k\theta$.......(1) [where k is proportionality constant and known as torsion constant of the wire/spring. The -ve sign indicates that the torque is restoring]

Also, torque of a rotating body can be expressed as

$\tau = I\alpha$.......(2) [where 'I' is moment of inertia of balance wheel about axis of rotation and α is the angular acceleration.]

From equations (1) and (2) we get,

$$I\alpha = -k\theta$$

$$\alpha = -\frac{k}{I}\theta \text{.......(3)}$$

This shows that $\alpha \propto$

Since angular acceleration is directly proportional to angular displacement so the *angular motion of balance wheel is simple harmonic motion.*

Time Period (T):

For angular simple harmonic motion acceleration is given by

$\alpha = -\omega^2\theta$.......(4)

From eqn (3) & (4) we get,

$$-\omega^2\theta = -\frac{k}{I}\theta$$

Or, $\omega^2 = \frac{k}{I}$

Or, $\omega = \sqrt{\frac{k}{I}}$

Or, $\frac{2\pi}{T} = \sqrt{\frac{k}{I}}$

$$\therefore T = 2\pi\sqrt{\frac{I}{k}}$$

This is the required expression for time period of balance wheel.

Expression for work done in terms of force constant and displacement in SHM

According to work energy theorem, work done is converted into P.E.

$\therefore$ W = P.E. $= \frac{1}{2} m\omega^2 y^2 = \frac{1}{2} m \frac{k}{m} y^2$ [Since $-\omega^2 y = \frac{-k}{m} y$]

$\therefore W = \frac{1}{2} k y^2$ (or, $W = \frac{1}{2} k x^2$)

Oscillatory Motion

To and fro periodic motion of a particle about a mean position is called an oscillatory motion in which a particle moves on either side of equilibrium (or) mean position is an oscillatory motion and $a \propto y^n$ where n = 1,3,5……. a=acceleration and y = displacement.

Also, we can write

$F = -k y^n$

→That means S.H.M. is the simplest type of oscillatory motion I.e. $a \propto y^1$ where n = 1

→It is a kind of periodic motion bounded between two extreme points. **For example,** Oscillation of Simple Pendulum, Spring-Mass System.

→The object will keep on moving between two extreme points about a fixed point is called mean position (or) equilibrium position along any path. (the path is not a constraint).

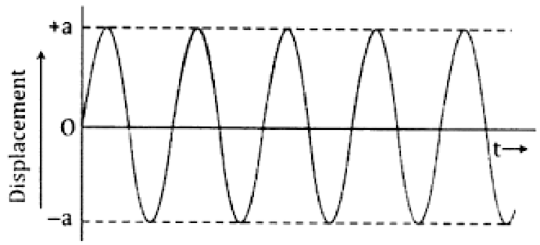
→There will be a restoring force directed towards equilibrium position (or) mean position.

→**In an oscillatory motion,** the net force on the particle is zero at the mean position.

→The mean position is a stable equilibrium position.

Free oscillation:

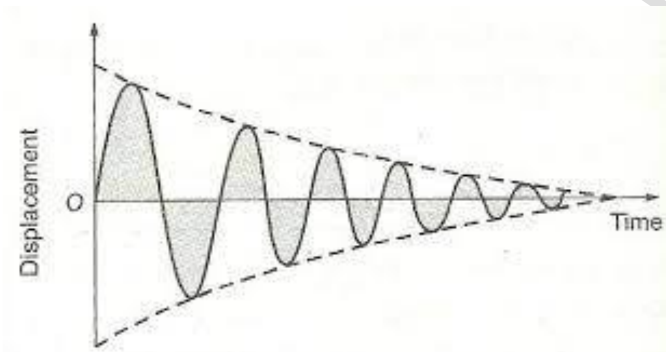
When a system or body capable of oscillation is given some initial displacement from its equilibrium position and left free, it begins to oscillate with its own natural frequency with the constant amplitude. Then the oscillation of the body is known as **free oscillation**.



The natural frequency of vibrations of the oscillating system depends upon inertia, elastic properties and dimension of the object. Oscillation of a pendulum, vibrating tuning fork or string all in vacuum are the examples of this type of motion. The vibration of electric and magnetic fields in an electromagnetic wave propagating in free space is the best possible example of this type.

Damped oscillation:

When the forces such as frictional or viscous force acts on a body executing SHM, the forces offer resistance to the motion. As a result, the mechanical energy of the body gradually decreases. The body therefore vibrates with gradually decreasing amplitudes and finally comes to the rest. This type of motion is known as damped oscillation. The motion of a pendulum in air/liquid, to and fro motion of a metallic strip in a magnetic field are examples of this type of oscillation. The variation of amplitude with time in damped oscillation is shown in the above figure.



Forced vibration and resonant vibration

If a body is set in vibration by an external periodic force then the oscillation is known as forced oscillation. When the frequency of forced oscillation is equal to the natural frequency of the vibrating body, the amplitude of vibration increases at each step and becomes very large. Such vibration is called resonant vibrations and the phenomenon resonance.

Resonance is a special case of forced vibration.